A Perturbation Solution Method for Models with Recursive Utility*

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Abstract

I illustrate a novel method for pricing assets within recursive utility models in continuous time, that has first been used in Melissinos (2023). My method builds on the analytic solution of Tsai and Wachter (2018). While their solution is valid for a value of the intertemporal elasticity of substitution, ψ, equal to 1, I provide the full perturbation series in terms of ψ, which gives rise to a global perturbation approximation. This allows the pricing of assets for a much larger range of values for ψ, which are economically meaningful. I comment on the convergence properties of the perturbation series, and I show that the method provides a straightforward and reliable approach to asset pricing. I employ my method to derive prices of long-term bonds, the price consumption ratio and the instantaneous return of the consumption perpetuity.

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1 Introduction

Even though utility functions with time-separable utility are most often used in the literature, recursive utility models are also popular, as they can describe more general preferences and behaviours. Indeed some models seem to require the use of recursive utility, for example Bansal and Yaron (2004) and Wachter (2013), in order to produce important features of the data (yyy). Recursive utility models in continuous time were introduced by Duffie and Epstein (1992b) and an important literature describing their properties has since followed, for example Duffie and Epstein 1992a; Duffie, Schroder and Skiadas 1996; Schroder and Skiadas 1999. However, solving these models in continuous time remains challenging. Recently, Tsai and Wachter (2018) suggested a method for pricing long-term assets using recursive utility in continuous time. In this paper, I introduce a perturbation method that is based on the analytical solution that Tsai and Wachter (2018) found.

The main underlying assumption of the method by Tsai and Wachter (2018) is that the intertemporal elasticity of substitution (IES) is equal to 1. While the authors claim that their solution can also be used for other IES values, it is not a priori obvious under which conditions this is true. In this paper I use their analytical solution as a base case and then perform a perturbation expansion around the IES value of 1. This allows me to get a solution that is valid for a large range of IES values and evaluate the accuracy of the approximation offered by Tsai and Wachter (2018). My approach is based on the perturbation theory described in Bender, Orszag and Orszag (1999), which has also been advanced in economics and finance by Judd (1996). Related literature includes Caldara, Fernandez-Villaverde, Rubio-Ramirez and Yao (2012), who solve DSGE models with recursive utility in discrete time. To the best of my knowledge, apart from Melissinos (2023) which uses the method described in this paper to solve long-run risk models and rare disaster models, Leisen (2016) is only one other paper that uses perturbation theory to solve recursive utility models in continuous time in a similar setup to what I examine here. However, unlike the current paper, Leisen (2016) looks at a model that also includes portfolio selection and the IES parameter is not the basis of the perturbation.

Once the value function is derived based on the perturbation approximation, I can derive the stochastic differential equation form of the stochastic discount factor (SDF), following the derivation of Chen, Cosimano, Himonas and Kelly (2009). Then I can proceed to solve the partial differential equation that is associated with the pricing of the long-term zero-coupon bond. In addition, quantities like the price-dividend ratio, the wealth-consumption ratio and the price of dividend-bearing securities can also be computed. I solve the pricing equation for the long-term bond by using the Feynman-Kac formula, which I implement through Monte Carlo simulations.

The rest of the paper is organised as follows. Section 2 introduces the general framework including the recursive utility component, Section 3 introduces the perturbation expansion, Section 4 performs the pricing of the long-term bond based on the previous results, and Section 5 concludes.

2 Recursive Utility Framework

This section closely follows the framework introduced by Tsai and Wachter (2018). For simplicity, I introduce my method using only one state variable. Introducing multiple state variables is straightforward based on the single-variable case.

2.1 Consumption process

The consumption process has two components: a deterministic trend and a Brownian motion component:[1]

$$\frac{dC_t}{C_t} = d\log(C_t) = dc_t = \mu_{ct}dt + \sigma_{ct}dZ_{ct} \tag{1}$$

where the t subscript denotes variables at time t,[2] C_t is consumption flow, c_t is log consumption flow, x_t is the state variable characterising the economy, μ_{ct} is the deterministic consumption trend, σ_{ct} determines consumption volatility

[1]Here, for simplicity, I assume that consumption does not undergo discontinuous jumps (with probability 1), but my solution method can be extended to the case, in which the consumption process includes Poisson jumps.

[2]For generality I use a subscript t for all symbols that can correspond to variables. However, in some variations these symbols may also correspond to parameters.

and Z_{ct} is the Brownian motion component. μ_{ct} and $\sigma_c t$ are either parameters or they depend on the state variable.

2.2 State variable

Similar to consumption, the state variable also has two components:

$$dx_t = \mu_{xt}dt + \sigma_{xt}dZ_{xt} \tag{2}$$

Here, x_t denotes the state variable, and the functions and parameters are analogous to the case of consumption.

2.3 Utility

Lifetime utility at time t_0 is:[3]

$$V_{t_0} = E_t\left[\int_{t_0}^{\infty} f(C_t, V_t)dt\right] \tag{3}$$

This equation highlights why utility is referred to as "recursive", as the integrand depends on the current value of the agent at each point in time. The combination of the consumption flow with the concurrent lifetime utility takes place via the so-called aggregator function:[4]

$$f(C, V) = \frac{(1-\gamma)\rho V\left(\left(C((1-\gamma)V)^{-\frac{1}{1-\gamma}}\right)^{1-\frac{1}{\psi}} - 1\right)}{1-\frac{1}{\psi}} \tag{4}$$

This function represents a flow which depends both on current consumption flow, C_t, and on the current level of the value, V_t. ρ denotes a time preference parameter, γ denotes the risk aversion parameter and ψ denotes the intertemporal elasticity of substitution (IES). Recursive utility is a useful modelling tool because it allows the separation of the risk aversion parameter from the IES parameter. This utility specification reduces to the more familiar time-additive

[3]Following Tsai and Wachter (2018) I do not prove existence and uniqueness of my solution. Hence, I use the infinite horizon case for simplicity. When considering a proof of existence and/or uniqueness, a finite horizon may be easier to deal with.

[4]This is the normalised form of the aggregator in Duffie and Epstein (1992b)

case when $\gamma = 1/\psi$. In this case the agent is indifferent about when uncertainty is resolved. For more general parameter specifications the agent may exhibit a preference for late or early resolution of uncertainty. In particular, for $\gamma > 1/\psi$ ($\gamma < 1/\psi$) there is a preference for early (late) resolution of uncertainty. The intuition for this mechanism can also be explained differently. In particular, $\gamma > 1/\psi$ implies that:

$$\frac{\partial^2 f(C,V)}{\partial C \partial V} < 0 \text{ and } \frac{\partial^3 f(C,V)}{\partial C^2 \partial V} > 0 \tag{5}$$

$$\Rightarrow E_t\left[\frac{\partial f(C,V_t)}{\partial C}\right] < E_t\left[\frac{\partial f(C,E_{t+1}[V_t])}{\partial C}\right] \tag{6}$$

On the right-hand side, the notation means that the agent has been given early information about the state of the world in $t+1$, while on the left-hand side this is not the case. It follows that the ex-ante expectation of these two situations leads to a preference for early resolution of uncertainty, as the utility flow is expected to be higher, in a state where the agent has early knowledge. The opposite is true in the case of a preference for late resolution of uncertainty. The mathematics of the situation is similar to the case, in which a risk-averse agent prefers a safe sum of money to a risky lottery with the same expected value. So, it is crucial for the preference of early resolution of uncertainty that consumption becomes less enjoyable as the value of the agent increases. Notice that this is not the familiar diminishing marginal utility of consumption. Instead, this implies that the same quantity of consumption is less enjoyable when the agent becomes happier for reasons that are not related to current consumption, for example, she may have learned that expected consumption in the future has increased and this makes her current consumption less enjoyable.

2.4 Decomposition of the value function

In the recursive utility framework, there exists a scale invariance property Duffie and Epstein (1992b), which allows us to express the value of the agent in a way that separates the dependence on consumption from the dependence on the state variable. As shown in Benzoni, Collin-Dufresne and Goldstein (2011) and

Tsai and Wachter (2018), the value function can be written as:

$$V = \frac{C^{1-\gamma} e^{K(x)(1-\gamma)}}{1-\gamma} \tag{7}$$

Where $K(x)$ satisfies the following differential equation:[5]

$$\rho \frac{\psi}{1-\psi}\left(1-e^{(1/\psi-1)K(x)}\right)-\frac{\gamma\sigma_{ct}^2}{2}+\mu_{ct}+\frac{\sigma_{xt}^2}{2}K''(x)+\mu_{xt}K'(x)+\frac{(1-\gamma)\sigma_{xt}^2}{2}K'(x)^2=0 \tag{8}$$

A proof of this result is included in Appendix A.1.[6]

2.5 Functional forms for consumption and state variable processes

The asset pricing problem based on the above setup is generally not easy to solve. However, Tsai and Wachter (2018) showed that significant progress can be made under the following specification for the consumption process and the process of the state variable:

$$\begin{aligned}
\mu_{ct} &= \mu_{c0} + \mu_{c1} x_t \\
\sigma_{ct} &= \sqrt{\sigma_{c0} + \sigma_{c1} x_t} \\
\mu_{xt} &= -\log(\phi)(\mu_{x0} - x_t) \\
\sigma_{xt} &= \sqrt{\sigma_{x0} + \sigma_{x1} x_t}
\end{aligned} \tag{9}$$

These parameters on the right hand side can be chosen. Notably, this specification is particularly useful because plugging these expressions into Equation (8) produces linear terms in x.

[5]This equation is also valid for $\psi = 1$, in which case the expressions are replaced by their limits.

[6]A very similar result is also proven by Tsai and Wachter (2018).

3 Method Description

3.1 Exact Solution for ψ=1

As Tsai and Wachter (2018) Equation 8 has an exact solution for $\psi = 1$. In particular, after the parametrisation from Expressions (9) is used, the differential equation becomes:

$$-\rho K(x) - \frac{1}{2}\gamma\left(\sigma_{\mathrm{c0}} + x\sigma_{\mathrm{c1}}\right) + \mu_{\mathrm{c0}} + x\mu_{\mathrm{c1}} + \frac{1}{2}K''(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) - \log(\phi)K'(x)\left(\mu_{\mathrm{x0}} - x\right) - \frac{1}{2}\gamma K'(x)^2\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) + \frac{1}{2}K'(x)^2\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) = 0 \tag{10}$$

and the solution takes the form, $K(x) = a_{0,0} + a_{0,1}x$. The coefficients, $a_{0,0}$ and $a_{0,1}$ can be solved by sequentially solving the following equations:

$$\begin{aligned} 0 =& -\rho a_{0,0} - \frac{1}{2}\gamma a_{0,1}^2\sigma_{\mathrm{x0}} - a_{0,1}\mu_{\mathrm{x0}}\log(\phi) + \frac{1}{2}a_{0,1}^2\sigma_{\mathrm{x0}} - \frac{\gamma\sigma_{\mathrm{c0}}}{2} + \mu_{\mathrm{c0}} \\ 0 =& -\rho a_{0,1} - \frac{1}{2}\gamma a_{0,1}^2\sigma_{\mathrm{x1}} + \frac{1}{2}a_{0,1}^2\sigma_{\mathrm{x1}} + a_{0,1}\log(\phi) - \frac{\gamma\sigma_{\mathrm{c1}}}{2} + \mu_{\mathrm{c1}} \end{aligned} \tag{11}$$

The second equation is solved first as it only includes parameter $a_{0,1}$. Then using this solution the first equation can also be solved.[7]

$$\begin{aligned} a_{0,1} =& -\frac{\rho - \log(\phi) \pm \sqrt{2(\gamma-1)\mu_{\mathrm{c1}}\sigma_{\mathrm{x1}} - (\gamma-1)\gamma\sigma_{\mathrm{c1}}\sigma_{\mathrm{x1}} + (\rho - \log(\phi))^2}}{(\gamma-1)\sigma_{\mathrm{x1}}} \\ & \text{or if } \sigma_{x1} = 0 \\ a_{0,1} =& \frac{2\mu_{\mathrm{c1}} - \gamma\sigma_{\mathrm{c1}}}{2\rho - 2\log(\phi)} \\ a_{0,0} =& -\frac{\gamma a_{0,1}^2\sigma_{\mathrm{x0}}}{2\rho} - \frac{a_{0,1}\mu_{\mathrm{x0}}\log(\phi)}{\rho} + \frac{a_{0,1}^2\sigma_{\mathrm{x0}}}{2\rho} - \frac{\gamma\sigma_{\mathrm{c0}}}{2\rho} + \frac{\mu_{\mathrm{c0}}}{\rho} \end{aligned} \tag{12}$$

Tsai and Wachter (2018) use this solution to derive analytical expressions for the pricing of long-term assets, when $\psi = 1$. They also use these results to derive approximate expressions for the case when $\psi \neq 1$.

[7] When $\sigma_{x1} \neq 0$, then $a_{0,1}$ has a double solution. However, only one of the two solutions is economically meaningful. This duplicity is explained by the existence of the square root and by the fact that the state variable can be defined to be an increasing or decreasing function of the state variable.

3.2 Extension of the method to the case $\psi \neq 1$

3.2.1 General description

In this paper, I extend the above solution method, in order to allow the parameter for IES to take a large range of values. As is common with perturbation solutions, instead of solving the problem for a specific value of ψ for which there is no analytic solution, the problem is redefined and solved for arbitrary ψ. This may seem as more difficult, but since the solution for $\psi = 1$ is already known, the perturbation solution provides a way to start from the solution that is known, and then gradually move towards a solution that is valid for any ψ. I achieve this by re-expressing ψ in terms of ϵ and expanding the problem in a series of ϵ. In particular, ψ is replaced by $\frac{1}{1-\epsilon}$ and the expansion of ψ in terms of ϵ is:

$$\psi = \frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \ldots \tag{13}$$

The redefinition in terms of ϵ is convenient because the analytic solution occurs for $\epsilon = 0$, and this makes the power series of $K(\cdot)$ considerably simpler. As above, I proceed by guessing the series solution of the differential equation (10):

$$\begin{aligned} K(x,\epsilon) =& \sum_{n=0}^{\infty} \epsilon^n \left(\sum_{m=0}^{n+1} a_{n,m} x^m \right) \\ =& \left(a_{0,0} + a_{0,1}x\right) \\ &+ \epsilon \left(a_{1,0} + a_{1,1}x + a_{1,2}x^2\right) \\ &+ \epsilon^2 \left(a_{2,0} + a_{2,1}x + a_{2,2}x^2 + a_{2,3}x^3\right) \\ &+ \epsilon^3 \left(a_{3,0} + a_{3,1}x + a_{3,2}x^2 + a_{3,3}x^3 + a_{3,4}x^4\right) \\ &\quad \ldots \\ =& K_0(x) + K_1(x)\epsilon + K_2(x)\epsilon^2 + \ldots \end{aligned} \tag{14}$$

In the remaining of the paper I refer to the approximations according to the highest power of ϵ. For example the approximation that only maintains the first line of Equation (14) is the "zeroth" order approximation. The approximation that maintains the first two lines is the "first" order approximation and so on. The structure of the solution is a polynomial both in terms of x and in terms of ϵ. In particular, for each successive order of ϵ the order of the polynomial in terms of x that is multiplying it increases by one. As it turns out, the solution of Tsai and Wachter (2018) corresponds to the zeroth order of the series. This makes sense given that $\epsilon = 0$ simplifies to the previous case, namely $\psi = 1$. The other higher polynomials in x are new and they will show what the effect is from moving away from IES equal to 1. Thus, by replacing ψ with $\frac{1}{1-\epsilon}$, then plugging in the guess and expanding in terms of ϵ, Equation (10) becomes:

$$
\begin{aligned}
0 = &-\frac{1}{2}\gamma\left(\sigma_{\mathrm{c0}}+x\sigma_{\mathrm{c1}}\right)+\mu_{\mathrm{c0}}+x\mu_{\mathrm{c1}}-\rho K_0(x)-\log(\phi)K_0'(x)\left(\mu_{\mathrm{x0}}-x\right)-\frac{1}{2}\gamma K_0'(x)^2\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)\\
&\qquad+\frac{1}{2}K_0'(x)^2\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)+\frac{1}{2}K_0''(x)\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)\Big)\\
&\epsilon\Big(-\rho K_1(x)-\log(\phi)K_1'(x)\left(\mu_{\mathrm{x0}}-x\right)-\gamma K_0'(x)K_1'(x)\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)+K_0'(x)K_1'(x)\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)\\
&\qquad+\frac{1}{2}K_1''(x)\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)\Big)\\
&\epsilon^2\Big(-\rho K_2(x)-\log(\phi)K_2'(x)\left(\mu_{\mathrm{x0}}-x\right)-\frac{1}{2}\gamma K_1'(x)^2\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)\\
&\quad-\gamma K_0'(x)K_2'(x)\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)+\frac{1}{2}K_1'(x)^2\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)+K_0'(x)K_2'(x)\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)\\
&\qquad+\frac{1}{2}K_2''(x)\left(x\sigma_{\mathrm{x1}}+\sigma_{\mathrm{x0}}\right)\Big)
\end{aligned}
\tag{15}
$$

In the expression above I have still not inserted the $a_{\cdot,\cdot}$ parameters in detail, in order to not clutter the overall expression too much. Nevertheless, it can be seen that for this equation to hold for all values of ϵ, we need the coefficient for each power of ϵ to be equal to 0. Subsequently, each of these coefficients includes the K_n's and their derivatives, which contain polynomials of x. Following the same strategy, for these polynomials to be equal to 0 for all values of x, the corresponding coefficients also need to equal 0. Combining the two stages, this implies that for each pair of powers for ϵ and x, that show up in Equation (14), there is a corresponding equation that allows us to compute the respective coefficient.[8] In addition, each of these equations turns out to be linear and sequentially solvable given the solutions of the previous equations.[9] The order for solving the equation increases with the powers of ϵ and decreases with the

[8]Another way to think of this is the following: For each n power of ϵ, there is a linear second order differential equation for $K_n(\cdot)$ which can be solved sequentially using the solutions of the previous differential equations.

[9]The only exception is parameter $a_{0,1}$ which was already mentioned above and might require the solution of a second order equation.

powers of x. Namely, the parameters are found in the following order:

$$a_{0,1}, a_{0,0}, a_{1,2}, a_{1,1}, a_{1,0}, a_{2,3}, a_{2,2}, a_{2,1}, a_{2,0}, \ldots \tag{16}$$

Given that each parameter requires the solution of a linear equation, it is easy to solve the model for high orders of approximation. However, for each order of ϵ the number of parameters increases by one and the equation become increasingly complicated. As a result, roughly fifteen orders of approximation in terms of ϵ can be found relatively quickly (this corresponds to 135 distinct parameter values).

3.2.2 First order approximation

Finding the first order approximation requires the solution of the following equations:

$$\begin{aligned}
0 &= -\rho a_{1,2} - 2\gamma a_{0,1}a_{1,2}\sigma_{\mathrm{x1}} + 2a_{0,1}a_{1,2}\sigma_{\mathrm{x1}} + 2a_{1,2}\log(\phi) \\
0 &= -\rho a_{1,1} - 2\gamma a_{0,1}a_{1,2}\sigma_{\mathrm{x0}} - 2a_{1,2}\mu_{\mathrm{x0}}\log(\phi) + 2a_{0,1}a_{1,2}\sigma_{\mathrm{x0}} - \gamma a_{0,1}a_{1,1}\sigma_{\mathrm{x1}} \\
&\qquad + a_{0,1}a_{1,1}\sigma_{\mathrm{x1}} + a_{1,2}\sigma_{\mathrm{x1}} + a_{1,1}\log(\phi) \\
0 &= -\rho a_{1,0} - \gamma a_{0,1}a_{1,1}\sigma_{\mathrm{x0}} - a_{1,1}\mu_{\mathrm{x0}}\log(\phi) + a_{0,1}a_{1,1}\sigma_{\mathrm{x0}} + a_{1,2}\sigma_{\mathrm{x0}}
\end{aligned} \tag{17}$$

whose solutions are:

$$\begin{aligned}
a_{1,2} &= \frac{\rho a_{0,1}^2}{2\left(2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)\right)} \\
a_{1,1} &= \frac{\rho a_{0,0}a_{0,1} - 2\gamma a_{1,2}a_{0,1}\sigma_{\mathrm{x0}} - 2a_{1,2}\mu_{\mathrm{x0}}\log(\phi) + 2a_{1,2}a_{0,1}\sigma_{\mathrm{x0}} + a_{1,2}\sigma_{\mathrm{x1}}}{\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)} \\
a_{1,0} &= \frac{\rho a_{0,0}^2 - 2\gamma a_{0,1}a_{1,1}\sigma_{\mathrm{x0}} - 2a_{1,1}\mu_{\mathrm{x0}}\log(\phi) + 2a_{0,1}a_{1,1}\sigma_{\mathrm{x0}} + 2a_{1,2}\sigma_{\mathrm{x0}}}{2\rho}
\end{aligned} \tag{18}$$

As can be seen above, the values of all parameters can be found by plugging in the previous solutions.

3.2.3 Second order approximation

The solution of the second order proceeds similarly. The equations to be solved are the following:

$$
\begin{aligned}
0 &= \frac{1}{6}\gamma\rho a_{0,0}^3 - \frac{1}{6}\rho a_{0,0}^3 - \gamma\rho a_{1,0}a_{0,0} + \rho a_{1,0}a_{0,0} + \frac{1}{2}\gamma^2\sigma_{\mathrm{x}0}a_{1,1}^2 - \gamma\sigma_{\mathrm{x}0}a_{1,1}^2 + \frac{1}{2}\sigma_{\mathrm{x}0}a_{1,1}^2 + \gamma\rho a_{2,0} \\
&\quad - \rho a_{2,0} + \gamma\log(\phi)\mu_{\mathrm{x}0}a_{2,1} - \log(\phi)\mu_{\mathrm{x}0}a_{2,1} + \gamma^2\sigma_{\mathrm{x}0}a_{0,1}a_{2,1} - 2\gamma\sigma_{\mathrm{x}0}a_{0,1}a_{2,1} + \sigma_{\mathrm{x}0}a_{0,1}a_{2,1} \\
&\quad - \gamma\sigma_{\mathrm{x}0}a_{2,2} + \sigma_{\mathrm{x}0}a_{2,2} \\
0 &= \frac{1}{2}\sigma_{\mathrm{x}1}a_{1,1}^2\gamma^2 + 2\sigma_{\mathrm{x}0}a_{1,1}a_{1,2}\gamma^2 + \sigma_{\mathrm{x}1}a_{0,1}a_{2,1}\gamma^2 + 2\sigma_{\mathrm{x}0}a_{0,1}a_{2,2}\gamma^2 - \sigma_{\mathrm{x}1}a_{1,1}^2\gamma + \frac{1}{2}\rho a_{0,0}^2a_{0,1}\gamma \\
&\quad - \rho a_{0,1}a_{1,0}\gamma - \rho a_{0,0}a_{1,1}\gamma - 4\sigma_{\mathrm{x}0}a_{1,1}a_{1,2}\gamma + \rho a_{2,1}\gamma - \log(\phi)a_{2,1}\gamma - 2\sigma_{\mathrm{x}1}a_{0,1}a_{2,1}\gamma \\
&\quad + 2\log(\phi)\mu_{\mathrm{x}0}a_{2,2}\gamma - \sigma_{\mathrm{x}1}a_{2,2}\gamma - 4\sigma_{\mathrm{x}0}a_{0,1}a_{2,2}\gamma - 3\sigma_{\mathrm{x}0}a_{2,3}\gamma + \frac{1}{2}\sigma_{\mathrm{x}1}a_{1,1}^2 - \frac{1}{2}\rho a_{0,0}^2a_{0,1} \\
&\quad + \rho a_{0,1}a_{1,0} + \rho a_{0,0}a_{1,1} + 2\sigma_{\mathrm{x}0}a_{1,1}a_{1,2} - \rho a_{2,1} + \log(\phi)a_{2,1} + \sigma_{\mathrm{x}1}a_{0,1}a_{2,1} - 2\log(\phi)\mu_{\mathrm{x}0}a_{2,2} \\
&\quad + \sigma_{\mathrm{x}1}a_{2,2} + 2\sigma_{\mathrm{x}0}a_{0,1}a_{2,2} + 3\sigma_{\mathrm{x}0}a_{2,3} \\
0 &= 2\sigma_{\mathrm{x}0}a_{1,2}^2\gamma^2 + 2\sigma_{\mathrm{x}1}a_{1,1}a_{1,2}\gamma^2 + 2\sigma_{\mathrm{x}1}a_{0,1}a_{2,2}\gamma^2 + 3\sigma_{\mathrm{x}0}a_{0,1}a_{2,3}\gamma^2 + \frac{1}{2}\rho a_{0,0}a_{0,1}^2\gamma \\
&\quad - 4\sigma_{\mathrm{x}0}a_{1,2}^2\gamma - \rho a_{0,1}a_{1,1}\gamma - \rho a_{0,0}a_{1,2}\gamma - 4\sigma_{\mathrm{x}1}a_{1,1}a_{1,2}\gamma + \rho a_{2,2}\gamma - 2\log(\phi)a_{2,2}\gamma \\
&\quad - 4\sigma_{\mathrm{x}1}a_{0,1}a_{2,2}\gamma + 3\log(\phi)\mu_{\mathrm{x}0}a_{2,3}\gamma - 3\sigma_{\mathrm{x}1}a_{2,3}\gamma - 6\sigma_{\mathrm{x}0}a_{0,1}a_{2,3}\gamma - \frac{1}{2}\rho a_{0,0}a_{0,1}^2 \\
&\quad + 2\sigma_{\mathrm{x}0}a_{1,2}^2 + \rho a_{0,1}a_{1,1} + \rho a_{0,0}a_{1,2} + 2\sigma_{\mathrm{x}1}a_{1,1}a_{1,2} - \rho a_{2,2} + 2\log(\phi)a_{2,2} + 2\sigma_{\mathrm{x}1}a_{0,1}a_{2,2} \\
&\quad - 3\log(\phi)\mu_{\mathrm{x}0}a_{2,3} + 3\sigma_{\mathrm{x}1}a_{2,3} + 3\sigma_{\mathrm{x}0}a_{0,1}a_{2,3} \\
0 &= \frac{1}{6}\gamma\rho a_{0,1}^3 - \frac{1}{6}\rho a_{0,1}^3 - \gamma\rho a_{1,2}a_{0,1} + \rho a_{1,2}a_{0,1} + 3\gamma^2\sigma_{\mathrm{x}1}a_{2,3}a_{0,1} - 6\gamma\sigma_{\mathrm{x}1}a_{2,3}a_{0,1} + 3\sigma_{\mathrm{x}1}a_{2,3}a_{0,1} \\
&\quad + 2\gamma^2\sigma_{\mathrm{x}1}a_{1,2}^2 - 4\gamma\sigma_{\mathrm{x}1}a_{1,2}^2 + 2\sigma_{\mathrm{x}1}a_{1,2}^2 + \gamma\rho a_{2,3} - \rho a_{2,3} - 3\gamma\log(\phi)a_{2,3} + 3\log(\phi)a_{2,3}
\end{aligned}
\tag{19}
$$

And the solutions are:

$$
\begin{aligned}
a_{2,3} =& -\frac{\rho a_{0,1}^3}{6\left(3\gamma a_{0,1}\sigma_{\mathrm{x1}} - 3a_{0,1}\sigma_{\mathrm{x1}} + \rho - 3\log(\phi)\right)} + \frac{\rho a_{1,2}a_{0,1}}{3\gamma a_{0,1}\sigma_{\mathrm{x1}} - 3a_{0,1}\sigma_{\mathrm{x1}} + \rho - 3\log(\phi)} \\
& - \frac{2\gamma a_{1,2}^2\sigma_{\mathrm{x1}}}{3\gamma a_{0,1}\sigma_{\mathrm{x1}} - 3a_{0,1}\sigma_{\mathrm{x1}} + \rho - 3\log(\phi)} + \frac{2a_{1,2}^2\sigma_{\mathrm{x1}}}{3\gamma a_{0,1}\sigma_{\mathrm{x1}} - 3a_{0,1}\sigma_{\mathrm{x1}} + \rho - 3\log(\phi)} \\
a_{2,2} =& -\frac{3a_{2,3}\mu_{\mathrm{x0}}\log(\phi)}{2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)} + \frac{3(1-\gamma)a_{2,3}a_{0,1}\sigma_{\mathrm{x0}}}{2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)} \\
& + \frac{2(1-\gamma)a_{1,2}^2\sigma_{\mathrm{x0}}}{2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)} - \frac{\rho a_{0,0}a_{0,1}^2}{2\left(2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)\right)} \\
& + \frac{\rho a_{1,1}a_{0,1}}{2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)} + \frac{2(1-\gamma)a_{1,1}a_{1,2}\sigma_{\mathrm{x1}}}{2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)} \\
& + \frac{\rho a_{0,0}a_{1,2}}{2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)} + \frac{3a_{2,3}\sigma_{\mathrm{x1}}}{2\gamma a_{0,1}\sigma_{\mathrm{x1}} - 2a_{0,1}\sigma_{\mathrm{x1}} + \rho - 2\log(\phi)} \\
a_{2,1} =& -\frac{2a_{2,2}\mu_{\mathrm{x0}}\log(\phi)}{\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)} + \frac{a_{2,2}\sigma_{\mathrm{x1}}}{\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)} \\
& + \frac{2(1-\gamma)a_{1,1}a_{1,2}\sigma_{\mathrm{x0}}}{\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)} + \frac{2(1-\gamma)a_{0,1}a_{2,2}\sigma_{\mathrm{x0}}}{\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)} \\
& + \frac{3a_{2,3}\sigma_{\mathrm{x0}}}{\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)} + \frac{(1-\gamma)a_{1,1}^2\sigma_{\mathrm{x1}}}{2\left(\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)\right)} \\
& - \frac{\rho a_{0,1}a_{0,0}^2}{2\left(\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)\right)} + \frac{\rho a_{1,1}a_{0,0}}{\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)} \\
& + \frac{\rho a_{0,1}a_{1,0}}{\gamma a_{0,1}\sigma_{\mathrm{x1}} - a_{0,1}\sigma_{\mathrm{x1}} + \rho - \log(\phi)} \\
a_{2,0} =& -\frac{a_{2,1}\mu_{\mathrm{x0}}\log(\phi)}{\rho} + \frac{(1-\gamma)a_{1,1}^2\sigma_{\mathrm{x0}}}{2\rho} + \frac{(1-\gamma)a_{0,1}a_{2,1}\sigma_{\mathrm{x0}}}{\rho} + \frac{a_{2,2}\sigma_{\mathrm{x0}}}{\rho} - \frac{1}{6}a_{0,0}^3 + a_{1,0}a_{0,0}
\end{aligned}
\tag{20}
$$

As can be seen above, the expressions become complicated fast. However, it is straightforward to use a computer to derive higher orders of approximation, at least up to the point that it is also too much for the computer.

3.3 Convergence

Regarding convergence the problem constitutes a regular perturbation problem (Bender *et al.* 1999). So, the series has a non-vanishing radius of converges around $\epsilon = 0$ for each x. This is known rigorously but it is not clear exactly what the radius of convergence is. Based on the definition of K in Equation

(14) some conclusions can be drawn. Firstly, a finite order of approximation will never work for all x. As x goes to $\pm\infty$, the highest power of x will necessarily blow up, in a way that does not correspond to the approximated function, as the highest power of x changes for each order of approximation. Furthermore, something can also be said regarding the convergence of the series. On the one hand, for small x, only parameters of the form $a_{n,0}$, $n = 0, 1, 2, 3, \ldots$ matter for the approximation. So, if their growth rate is slower than the decay rate of ϵ^n, then the approximation will converge. On the other hand, if x is not very small, then the parameters of the form $a_{n,n+1}$, $n = 0, 1, 2, 3, \ldots$ will matter for convergence, and the approximation will converge, if the growth rate of these parameters is slower than the decay rate of $(\epsilon \times x)^n$. So, then the question is whether we can deduce the growth rate of these parameters.

Indeed the parameters follow some patterns, that can already be seen in the expression that I provided in the previous subsection. Firstly, the parameters of the form $a_{n,n+1}$, with $n = 0, 1, 2, \ldots$ are determined recursively based on other parameters of the same form, $a_{n',n'+1}$, with $n = 0, 1, 2, \ldots$ and $n' < n$. Thus, these "diagonal" parameters can be computed independently. In addition, the pattern of products implies that any parameter of the form $a_{n,n+1}$ for each n includes terms containing $a_{0,1}$ raised at most to the power $n + 1$. In fact, something similar holds for all parameters of the nth order approximation. In particular, when a parameter $a_{n,m}$ is written in terms of $a_{0,0}$ and $a_{0,1}$, each term in the corresponding expression contains combinations of powers of these two initial parameters, and the sum of the powers is always less or equal to $n + 1$. This can be seen, for instance, in Figure 1.[10]

This indicates that as long as the following conditions jointly hold the series will converge:

- ϵ^n decays faster than the growth of $a_{0,1}^{(n+1)}$
- ϵ^n decays faster than the growth of $a_{0,0}^n$
- $(\epsilon x)^n$ decays faster than the growth of $a_{0,1}^{n+1}$

[10]Here I set $\gamma = 1$. This facilitates some simplifications. In the general case and as long as the exact growth properties are considered an extra factor would arise, as in each step going from $m + 1$ to 0 there are terms that contain an extra power of $a_{0,1}$ both in the numerator and in the denominator. This would affect the third case that I show below.

	1	x	x^2	x^3	x^4	x^5
1	$a_{0,0}$	$a_{0,1}$	0	0	0	0
ϵ	$0.+0.5\,a_{0,0}^2+0.806542\,a_{0,0}\,a_{0,1}+0.360086\,a_{0,1}^2$	$0.+0.193458\,a_{0,0}\,a_{0,1}+0.0863705\,a_{0,1}^2$	$0.+0.0535437\,a_{0,1}^2$	0	0	0
ϵ^2	$0.+0.333333\,a_{0,0}^3+0.962574\,a_{0,0}^2\,a_{0,1}+0.898055\,a_{0,0}\,a_{0,1}^2+0.271482\,a_{0,1}^3$	$0.+0.037426\,a_{0,0}^2\,a_{0,1}+0.129038\,a_{0,0}\,a_{0,1}^2+0.0651179\,a_{0,1}^3$	$0.-0.0270929\,a_{0,0}\,a_{0,1}^2-0.00196799\,a_{0,1}^3$	$0.-0.00837498\,a_{0,1}^3$	0	0
ϵ^3	$0.+0.25\,a_{0,0}^4+0.99276\,a_{0,0}^3\,a_{0,1}+1.46067\,a_{0,0}^2\,a_{0,1}^2+0.936459\,a_{0,0}\,a_{0,1}^3+0.219814\,a_{0,1}^4$	$0.+0.00724036\,a_{0,0}^3\,a_{0,1}+0.056934\,a_{0,0}^2\,a_{0,1}^2+0.0898408\,a_{0,0}\,a_{0,1}^3+0.0371744\,a_{0,1}^4$	$0.-0.0176065\,a_{0,0}^2\,a_{0,1}^2-0.0332772\,a_{0,0}\,a_{0,1}^3-0.0129955\,a_{0,1}^4$	$0.-0.000645249\,a_{0,0}\,a_{0,1}^3-0.00244501\,a_{0,1}^4$	$0.+0.00044994\,a_{0,1}^4$	0
ϵ^4	$0.+0.2\,a_{0,0}^5+0.998599\,a_{0,0}^4\,a_{0,1}+1.98805\,a_{0,0}^3\,a_{0,1}^2+1.96372\,a_{0,0}^2\,a_{0,1}^3+0.958123\,a_{0,0}\,a_{0,1}^4+0.184172\,a_{0,1}^5$	$0.+0.00140071\,a_{0,0}^4\,a_{0,1}+0.0182958\,a_{0,0}^3\,a_{0,1}^2+0.0591872\,a_{0,0}^2\,a_{0,1}^3+0.0625563\,a_{0,0}\,a_{0,1}^4+0.0207276\,a_{0,1}^5$	$0.-0.00634651\,a_{0,0}^3\,a_{0,1}^2-0.0311549\,a_{0,0}^2\,a_{0,1}^3-0.0346041\,a_{0,0}\,a_{0,1}^4-0.0113813\,a_{0,1}^5$	$0.+0.00403845\,a_{0,0}^2\,a_{0,1}^3+0.00259804\,a_{0,0}\,a_{0,1}^4+0.000192267\,a_{0,1}^5$	$0.+0.00136972\,a_{0,0}\,a_{0,1}^4+0.000603844\,a_{0,1}^5$	$0.+0.000153175\,a_{0,1}^5$

$\{\gamma=1,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\,x_t},\ \sigma_{x1}=0.0169,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 1: Series Coefficients – Variation: Time-varying consumption volatility.

This shows the value of the parameters in terms of $a_{0,0}$ and $a_{0,1}$. The first row and first column show the corresponding power of x and ϵ respectively. The nth power of ϵ and mth power of x correspond to $a_{n,m}$. It can be seen that the highest power of $a_{0,0}$, $a_{0,1}$ or the higher sum of their powers for the parameters in the nth order approximation is $n+1$. The calibration used is also labeled.

Even though these conditions are mostly reliable, they are not exact, as the series can converge when the conditions are violated, and it can also diverge when the conditions hold. In order to prove the exact conditions for convergence, a more extensive analysis needs to be made of the combinatorial structure of the problem, so that the growth rate of the coefficients can be exactly determined. However, in practise these conditions are good indications regarding convergence, which can practically be checked by looking at the first partial sums of the series. Figure 2 shows these partial sums for the same calibration as in Figure 1. The top plot shows convergence of the series for $K(x)$ for different values of x according to the approximate conditions expressed previously, the series should converge as long as $|x \times \epsilon| < 1/a_{0,1}$, that is less than 238 approximately. In fact, the series seems to converge for much larger values also, and it starts diverging when $|x \times \epsilon|$ is about 1200. These numbers are huge, as in this calibration the state variable would practically never take values larger than 10, which would mean that consumption volatility is ten times larger compared to the steady state. The bottom plot shows the convergence of the series for the derivative of $K(x)$ for different values of x, and it is clear that the convergence of the derivative follows the same pattern. This is reasonable given that in this case convergence is regulated by the terms that have a high power in terms of x, and these terms appear in both K and its derivative.

Figure 3 shows convergence for different values of ϵ using the same calibration.[11] According to the approximate conditions, the series should converge, if the absolute value of ϵ is less than 0.8. Indeed, as can be seen in the figure the series begins to diverge both for $\epsilon = 0.8$ and for $\epsilon = -0.8$. The figure also shows the corresponding values of ψ for each value of ϵ, and in this example the figurer indicates that the zeroth order approximation used by Tsai and Wachter (2018) can indeed be used for a significant range of ψ values. However, using higher orders of approximation leads to a much larger range of ψ values becoming usable. The second and third plots of Figure 3 show respectively that the first and second derivative of K converge for all the values of ϵ, that I have chosen, including the values of ϵ for which K itself diverged. This can be explained, because the convergence of the series depends on the terms containing the lowest

[11] Apart from γ which is now equal to 2.

powers of x, as terms with higher powers of x will almost certainly be smaller. Hence, the derivatives are much more likely to converge compared to the original function. This means that for some quantities that only rely on the derivatives, the approximation may be used even when K itself diverges.

4 Pricing

4.1 Process of the stochastic discount factor

As has been shown already the method can provide reliable approximations for a large range of ψ values. The next step is to use the approximation to perform the pricing of securities. This requires the derivation of the process of the SDF. In particular, given the expression for the value function, Ito's Lemma can be implemented to get to the stochastic differential equation that governs the SDF. The calculation here follows Chen *et al.* (2009). In particular, the fundamental relationship is:

$$\frac{d\Lambda}{\Lambda} = f_V(C,V)dt + \frac{df_C(C,V)}{f_C(C,V)} \tag{21}$$

This can be computed relatively easily. The first term is just the derivative of the flow utility with respect to the value function. The second term can be computed with an application of Ito's lemma on the derivative of flow utility with respect to consumption.[12] The result is the following:

$$\begin{aligned}\frac{d\Lambda}{\Lambda} =& \Bigg(\frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi\right)}{1-\psi} - \gamma\mu_{ct} + \frac{\gamma^2\sigma_{ct}^2}{2} + \frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\ &+ \frac{(\gamma\psi-1)\Big(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2\left((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t)\right)\Big)}{2\psi^2}\Bigg)dt \\ &+ \frac{(1-\gamma\psi)\sigma_{xt}K'(x_t)}{\psi}dZ_{xt} - \gamma\sigma_{ct}dZ_{ct}\end{aligned} \tag{22}$$

where ρ_{cx} is the correlation between consumption and the state variable. The time-separable case arises when $\gamma = 1/\psi$. As can be seen by the expression

[12] It is possible to do this operation after substituting the value function using Equation (7) and applying Ito's lemma based on consumption and the state variable as independent variables.

above the dependence of the SDF on $K(x)$ and on x disappears in this case. The stochastic component that is related to consumption ($-\gamma\sigma_{ct}dZ_{ct}$) is the same as in the time-separable case. On the contrary, the stochastic component that is related to the state variable ($(1-\gamma\psi)\sigma_{xt}K'(x_t)/\psi dZ_{xt}$) does not appear in the time-separable case. So, given that these stochastic components are ultimately responsible for the generation of risk premia, recursive utility introduces an extra mechanism by which premia can be generated. The sign of this mechanism depends on the sign of $(1-\gamma\psi)$, which corresponds to a preference for late or early resolution of uncertainty.

Furthermore, based on the SDF expression the risk-free rate can also be deduced:

$$\begin{aligned} r(x) = & -E\frac{d\Lambda}{\Lambda}\frac{1}{dt} = \\ & \gamma\mu_{ct} - \frac{\gamma^2\sigma_{ct}^2}{2} + \frac{\rho\left((1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} + \gamma\psi - \psi\right)}{1-\psi} + \frac{\gamma(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\ & + \frac{(1-\gamma\psi)\Big(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2\left((1-\gamma\psi)K'(x_t)^2 - \psi K''(x_t)\right)\Big)}{2\psi^2} \end{aligned} \tag{23}$$

The short rate is also affected by recursive utility. While the consumption smoothing motive ($\gamma\mu_{ct}$) and the precautionary savings motive ($-\frac{\gamma^2\sigma_{ct}^2}{2}$) is the same as in the time-separable case, the time preference parameter is multiplied by a new factor, and the remaining terms are all new.

4.2 Long-term bonds

The process for the SDF can be inserted in the pricing differential equation as in Cochrane (2009) and Chen *et al.* (2009):

$$E\big[d(\Lambda Q)\big] = 0 \Rightarrow E\left[\frac{d\Lambda}{\Lambda} + \frac{dQ}{Q} + \frac{d\Lambda dQ}{\Lambda Q} = 0\right] = 0 \tag{24}$$

Here $Q(m,x)$ is the price of the zero-coupon bond with maturity m when the state of the economy is x.[13] By Ito's Lemma:

$$dQ(x,m) = \big(-\log(\phi)(\mu_{x0} - x)Q_x - Q_m + \frac{1}{2}\sigma_x^2 Q_{xx}\big)dt + \sigma_{xt}Q_x dZ_{xt} \tag{25}$$

This can be directly plugged in Equation 24 and the result is:

$$\begin{aligned} 0 =& - Q_m + r(x)Q + \left(-\log(\phi)(\mu_{x0} - x) + A(x)\right) Q_x + \frac{1}{2}Q_{xx}\sigma_{xt}^2 \\ A(x) =& \; (\gamma + \epsilon - 1)\sigma_{xt}^2 K^{(1,0)}(x,\epsilon) + \gamma\rho_{cx}\sigma_{ct}\sigma_{xt} \end{aligned} \tag{26}$$

The subscripts $\cdot_m$ and $\cdot_x$ denote partial derivatives with respect to maturity, m, and with respect to the state variable, x, respectively. In the above equation $K(x)$ appears in $r(x)$, but the coefficients of Q_x and Q_{xx} only contain $K^{(1,0)}(x)$. This is noteworthy because Equations (25) and (23)imply that the expected instantaneous excess return obeys the following relationship:

$$E\left[\frac{dQ}{Q}\right] - r(x_t)dt = -E\left[\frac{d\Lambda dQ}{\Lambda Q}\right] = A(x)dt = \Big((\gamma+\epsilon-1)\sigma_{xt}^2 K^{(1,0)}(x,\epsilon) + \gamma\rho_{cx}\sigma_{ct}\sigma_{xt}\Big)dt \tag{27}$$

So, the term premium also primarily depends on $K'(x)$ and not $K(x)$ itself. This implies that my approximation may be able to provide useful information about term premia even when it diverges, given the result in Section 3.3 that the derivative of K can converge even when K diverges.[14]

Continuing with the pricing of the long-term bond, according to the Feynman-Kac method Equation (37) can be solved by Monte Carlo simulations. In particular:

$$Q(m, x_t) = E\left[\exp\left\{\int_m^0 r(\tilde{x}_{t+s})ds\right\}\right] = E\left[\exp\left\{-\int_0^m r(\tilde{x}_{t+s})dt\right\}\right] \tag{28}$$

[13] In the formulas I use Q instead of $Q(m,x)$ to avoid cluttering.

[14] For brave researchers this could suggest the use of this approximation even when the original series diverges, when the item of interest is the risk premium, which is determined by the derivative of $K(x)$.

where $\tilde{x}_0 = x$ and $\tilde{x}_t$ follows the modified process:

$$d\tilde{x} = \big(-\log(\phi)(\mu_{x0} - \tilde{x}) + (\gamma + \epsilon - 1)\sigma_x(\tilde{x})^2 K'(\tilde{x}) + \gamma\rho_{cx}\sigma_c(\tilde{x})\sigma_x(\tilde{x})\big)\, dt + \sigma_x(\tilde{x}_t) dZ_{xt} \tag{29}$$

This is a modified process because, while it is similar to the regular state variable of the model, the trend component of the modified process has extra terms coming from the interaction of the SDF with the stochastic components of the state variable process.[15] Based on function Q and Equation (25), it is also easy to derive the instantaneous expected return of long-term bonds.

If, instead of using the modified process, the original state variable is used:

$$H(m, x_t) = E\left[\exp\left\{\int_m^0 r(x_{t+s})ds\right\}\right] = E\left[\exp\left\{-\int_0^m r(x_{t+s})dt\right\}\right] \tag{30}$$

The result is the price of the *risk-neutral bond*, namely a bond priced by a risk-neutral investor with the same consumption process and utility function as in the original model. The difference between the yields of Q and H can be defined as the term premium for the corresponding maturity:

$$TP(m, x_t) = -\frac{\log\big(Q(m, x_t)\big)}{m} - \left(-\frac{\log\big(H(m, x_t)\big)}{m}\right) \tag{31}$$

4.3 Price-consumption ratio

The price consumption-ratio is a concept similar to the price-dividend ratio. It is a ratio, whose numerator is the price of the consumption perpetuity, a security that continuously pays the consumption flow for an infinite horizon, and its denominator is the concurrent consumption flow. Wachter (2006) derived the price consumption ratio in discrete time for the habit model of Campbell and Cochrane (1999). Here I use the same approach adapted for continuous time. So, I build up the price-consumption ratio from zero-coupon securities that pay the value of consumption after m periods. These securities have a price $P(m, x_t, C_t)$ at time t. The value of these securities depends on the current value of the sate variable and the current value of consumption. In order to avoid the dependence

[15] In this expression, instead of σ_{ct} and σ_{xt}, I am using $\sigma_c(\tilde{x})$ and $\sigma_x(\tilde{x})$ in order to make explicit that these quantities can now be functions of the modified process $\tilde{x}$.

on consumption, I divide these securities by current consumption. This leads to a *zero-coupon m-year price consumption ratio*, $q(m,x) = P(m,x,C_t)/C_t$. Then the combination of these zero-coupon securities leads to the full price-consumption ratio:[16]

$$p(x_t) = \int_0^\infty q(m,x_t)dm = \int_0^\infty \frac{P(m,x_t,C_t)}{C_t}dm \tag{32}$$

Furthermore, one can also define price-consumption *annuity* ratio. The consumption annuity is similar to the consumption perpetuity, but it only pays coupons for a finite period of time M. For example:

$$p_M(x_t) = \int_0^M q(m,x_t)dm = \int_0^M \frac{P(m,x_t,C_t)}{C_t}dm \tag{33}$$

If M is large, this quantity likely behaves similar to the price-consumption ratio, but in practice this may be easier to compute as it does not require the calculation of the integral for an infinite horizon.

Moving on, for simplicity I drop the time subscript in the following expressions. In order to derive $q(m,x)$, I follow an approach similar to Chen, Cosimano and Himonas (2010), who use the pricing equation to calculate the price-consumption ratio directly. Unlike them I first calculate the q's and I then build up the price-consumption ratio. This is arguably more complicated as it involves the solution of a partial differential equation and the computation of an integral, instead of the solution of an ordinary differential equation only. However, my approach does not require the specification of initial conditions and it determines the price-consumption ratio uniquely. Thus, the pricing equation can be re-written:

$$\begin{aligned} E\big[d(\Lambda P(m,x,C))\big] = 0 &\Rightarrow E\left[\frac{d\Lambda}{\Lambda} + \frac{dP(m,x,C)}{P(m,x,C)} + \frac{d\Lambda dP(m,x,C)}{\Lambda P(m,x,C)}\right] = 0 \\ &\Rightarrow E\left[\frac{d\Lambda}{\Lambda} + \frac{d\Big(q(m,x)C\Big)}{q(m,x)C} + \frac{d\Lambda d\Big(q(m,x)C\Big)}{\Lambda q(m,x)C}\right] = 0 \\ &\Rightarrow E\left[\frac{d\Lambda}{\Lambda} + \frac{dq}{q} + \frac{dC}{C} + \frac{d\Lambda dq}{\Lambda q} + \frac{d\Lambda dC}{\Lambda C} + \frac{dqdC}{qC}\right] = 0 \end{aligned} \tag{34}$$

[16] This assumes that the integral is finite.

In the final line I do not show the dependence of p for simplicity. Similar to above, by Ito's Lemma:

$$dq = \big(-\log(\phi)(\mu_{x0} - x)q_x - q_m + \frac{1}{2}\sigma_x^2 q_{xx}\big)dt + \sigma_{xt}q_x dZ_{xt} \tag{35}$$

So the processes for the SDF, for the zero-coupon consumption security and for consumption can all be substituted in the equation above and this will again generate a partial differential equation that can be solved, by computing the Feynman-Kac formula through Monte Carlo simulations. The pricing equation is:

$$\begin{aligned} 0 &= \underbrace{-r(x)}_{d\Lambda/\Lambda} + \underbrace{\Big(-\log(\phi)(\mu_{x0} - x)\frac{q_x}{q} - \frac{q_m}{q} + \frac{1}{2}\frac{q_{xx}}{q}\sigma_{xt}^2\Big)}_{dq/q} + \underbrace{\mu_{ct}}_{dC/C} \\ &\quad + \underbrace{\frac{(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} - \gamma\sigma_{ct}^2}_{d\Lambda dC_t/(\Lambda C_t)} + B(x)\frac{q_x}{q} \\ B(x) &= \underbrace{\frac{(1-\gamma\psi)\sigma_{xt}^2 K'(x_t)}{\psi} - \gamma\rho_{cx}\sigma_{xt}\sigma_{ct}}_{d\Lambda dq/(\Lambda q)} + \underbrace{\rho_{cx}\sigma_{xt}\sigma_{ct}}_{dqdC_t/(qC_t)} \end{aligned} \tag{36}$$

The brackets show where the expressions in the equations come from. The equation can be rewritten as:

$$\begin{aligned} 0 = &-\left(r(x) - \mu_{ct} - \frac{(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} + \gamma\sigma_{ct}^2\right)q - q_m + \frac{\sigma_{xt}^2}{2}q_{xx} \\ &+\Big(-\log(\phi)(\mu_{x0} - x) + \frac{(1-\gamma\psi)\sigma_{xt}^2 K'(x_t)}{\psi} - \gamma\rho_{cx}\sigma_{xt}\sigma_{ct} + \rho_{cx}\sigma_{xt}\sigma_{ct}\Big)q_x \end{aligned} \tag{37}$$

The corresponding Feynman-Kac formula is:

$$q(m, x_t) = E\left[\exp\left\{\int_m^0 \bar{r}(\bar{x}_{t+s})ds\right\}\right] = E\left[\exp\left\{-\int_0^m \bar{r}(\bar{x}_{t+s})dt\right\}\right] \tag{38}$$

where

$$\bar{r}(\bar{x}) = r(\bar{x}) - \mu_c(\bar{x}) - \frac{(1-\gamma\psi)\rho_{cx}\sigma_x(\bar{x})\sigma_c(\bar{x})K'(\bar{x})}{\psi} + \gamma\sigma_c(\bar{x})^2 \tag{39}$$

$\bar{x}_0 = x_0$ and $\bar{x}$ follows another modified process:[17]

$$d\bar{x} = \left(-\log(\phi)(\mu_{x0} - \bar{x}) + \frac{(1-\gamma\psi)\sigma_x(\bar{x})^2 K'(x_t)}{\psi} + (1-\gamma)\rho_{cx}\sigma_x(\bar{x})\sigma_c(\bar{x}) \right) dt + \sigma_x(\bar{x}_t) dZ_{xt} \tag{40}$$

Given the price-consumption ratio as a function of the state variable and the given stochastic process of the price-consumption ratio that can be written as follows:

$$dp = \big(-\log(\phi)(\mu_{x0} - x)p_x + \frac{1}{2}\sigma_x^2 p_{xx} \big) dt + \sigma_{xt} p_x dZ_{xt} \tag{41}$$

The return of the consumption perpetuity can be derived:[18]

$$\begin{aligned} \frac{dP}{P} + \frac{C}{P}dt =& \frac{d(Cp)}{Cp} + \frac{1}{p} = \frac{dC}{C} + \frac{dp}{p} + \frac{dCdp}{Cp} + \frac{1}{p}dt \\ =& \mu_{ct}dt + \sigma_{ct}dZ_{ct} - \log(\phi)(\mu_{x0} - x)\frac{p_x}{p}dt + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p}dt \end{aligned} \tag{42}$$

$$+ \sigma_{xt}\frac{p_x}{p}dZ_{xt} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p}dt + \frac{1}{p}dt$$

$$= \left(\mu_{ct} - \log(\phi)(\mu_{x0} - x)\frac{p_x}{p} + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p} + \frac{1}{p} \right) dt + \sigma_{ct}dZ_{ct} + \sigma_{xt}\frac{p_x}{p}dZ_{xt}$$

Finally the expected return is:

$$E\left[\frac{dP}{P}\right] + \frac{C}{P}dt = \left(\mu_{ct} - \log(\phi)(\mu_{x0} - x)\frac{p_x}{p} + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p} + \frac{1}{p} \right) dt \tag{43}$$

5 Applications

5.1 Time-varying consumption drift

Given the setup introduced in the previous section, real interest rates and the price consumption ratio can be determined. Here, I show the results for the case when consumption drift is time-varying. This is the result of setting $\mu_{c1} = 1$,

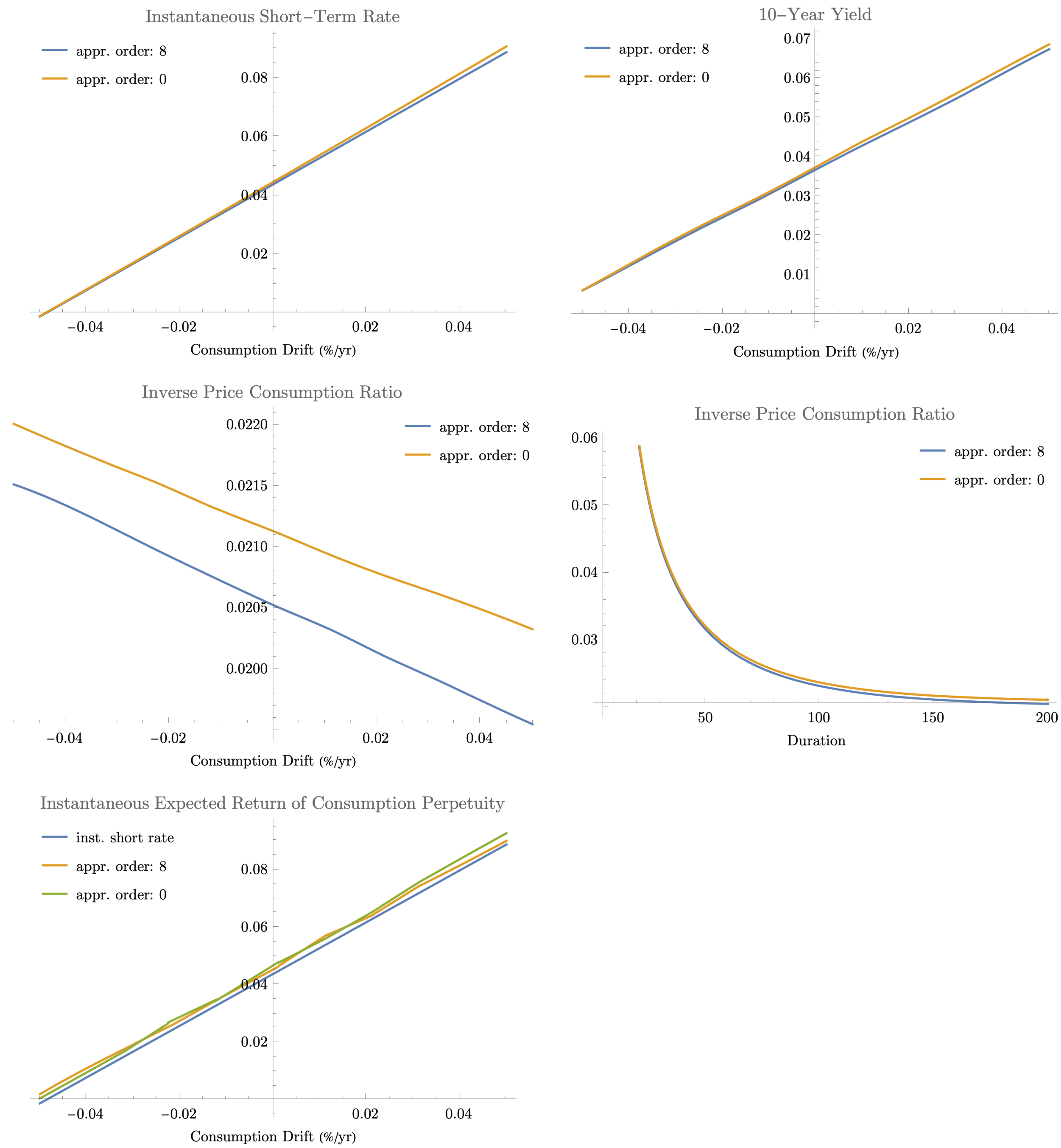
[17] Similar to above, in this expression, instead of σ_{ct} and σ_{xt}, I am using $\sigma_c(\bar{x})$ and $\sigma_x(\bar{x})$ in order to make explicit that these quantities can now be functions of the modified process $\bar{x}$.

[18] This calculation only applies for the consumption perpetuity. For the case of the consumption annuity the calculation would require an extra component that accounts for the fact that the annuity at time t has infinitesimally lower duration compared to the annuity at time $t + dt$. Numerically, I only calculate annuities. So, I only apply this formula for long-lived annuities, for which the security's price should not change significantly with duration.

which means that the consumption drift is proportional to the state variable. In this variation consumption volatility is constant, because $\sigma_{c1} = 0$, and the process is homoskedastic, because $\sigma_{x1} = 0$. Figure 4 shows the results, while comparing the eighth order approximation, using the method introduced in this paper, to the zeroth order approximation, that is equivalent to the method of Tsai and Wachter (2018). In addition, here $\epsilon = 0.1$ ($\psi = 1.11$) which is close to the analytic solution for $\epsilon = 0$ ($\psi = 1$). The results verify that the basic approximation can be accurate for $\psi \neq 1$. The fist row shows the instantaneous rate and the ten-year yield as a function of the state of the economy, which is reflected by consumption drift. The results for the two approximations are very similar. The second row shows the inverse price consumption ratio as a function of the consumption drift. The inverse price consumption ratio is equivalent to the dividend yield of the security. As I have already mentioned, I calculate the price consumption ratio numerically by integrating the zero coupon price consumption ratios. However, I cannot numerically integrate to infinity. So, I put the cutoff at 200 years. This means that technically I am calculating price consumption ratio for the 200 year consumption annuity. The second plot of the second row shows the value of the inverse price consumption ratio for different cutoff points and it can be seen that at 200 hundred years it is relatively close to being converged. Similar to above, the price consumption ratio for the two approximations is quite close, even though in this case the price consumption ratio is not very sensitive to the value of consumption drift. So, the difference appears larger in the figure. Finally in the third row, I also show the instantaneous expected return of the consumption perpetuity, which is very similar to the instantaneous short-term rate in this variation.

In Figure 5, I show the results for $\epsilon = 0.7$, which is not so close to the analytic solution of $\epsilon = 0$, and as can be seen in Figure 9, the value function varies significantly for different orders of approximation. This example illustrates the value of my method, as it shows that for some interesting values for the intertemporal elasticity of substitution ($\psi = 3.3$ in this case, but in other examples it can also be lower), the value function deviates significantly between the different orders of approximation. And this has consequences for the implied value of the short-term rate, the yield and the price consumption ratio.

The short-term rate and the ten-year yield appear linear as functions of the consumption drift for both approximations, but the slope is different and at the steady state there is a difference of roughly 1% for the short-term rate and a bit lower for the ten-year yield. The difference between the two approximations is also higher for the price consumption ratio, which is now also more sensitive to the consumption drift.



$\{\gamma=2, \rho=0.022, \phi=0.92, \rho_{cx}=0.5, \mu_{c0}=0.0252, \mu_{c1}=1, \mu_{x0}=0, \sigma_{c0}=0.0004, \sigma_{c1}=0, \sigma_{x0}=0.000159883, \sigma_{x1}=0\}$

Figure 4: Variation: Time-varying consumption drift $-$ $\epsilon = 0.1$
The first row shows the .

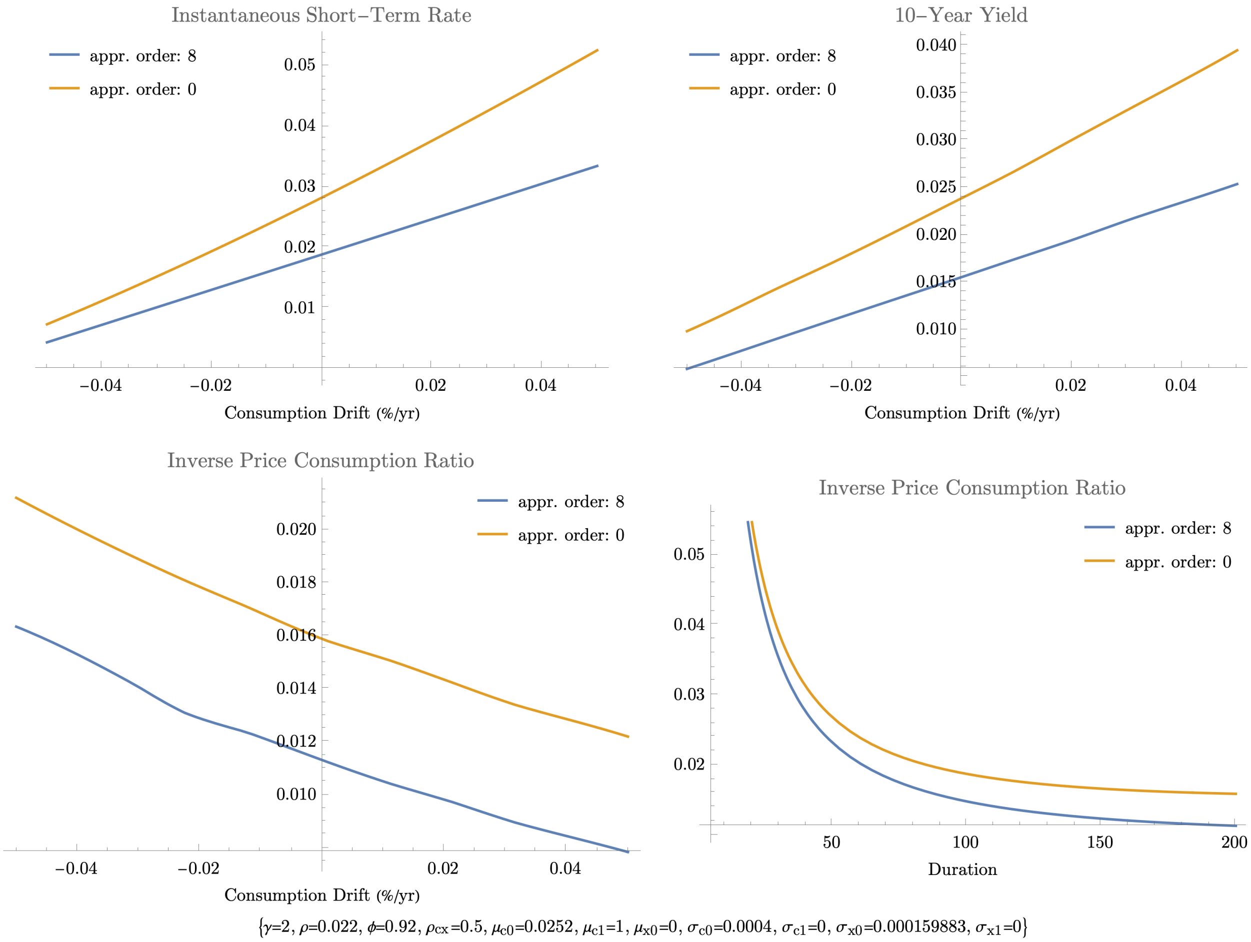
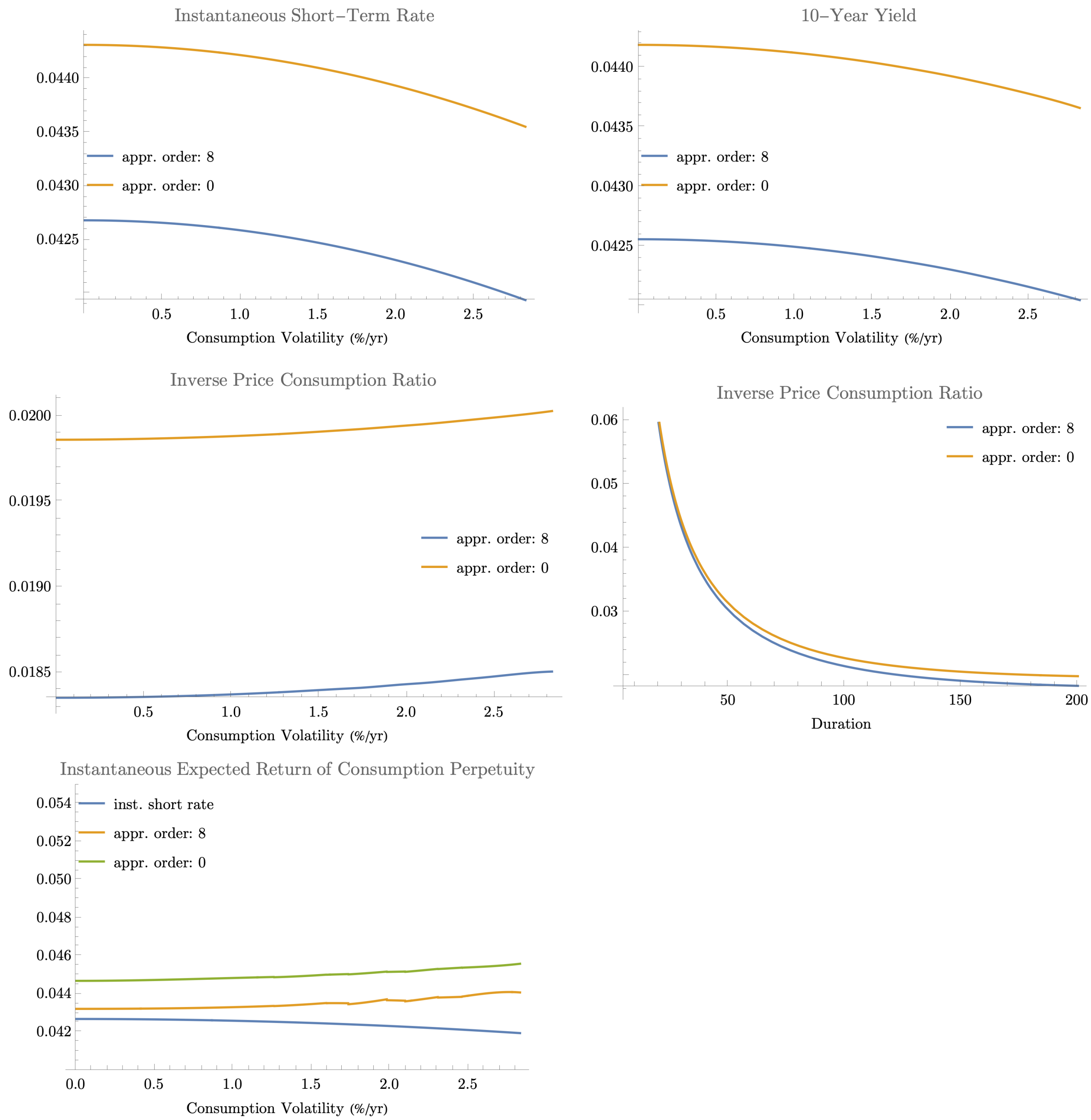


Figure 5: Variation: Time-varying consumption drift $-\ \epsilon = 0.1$
The first row shows the .

5.2 Time-varying consumption volatility

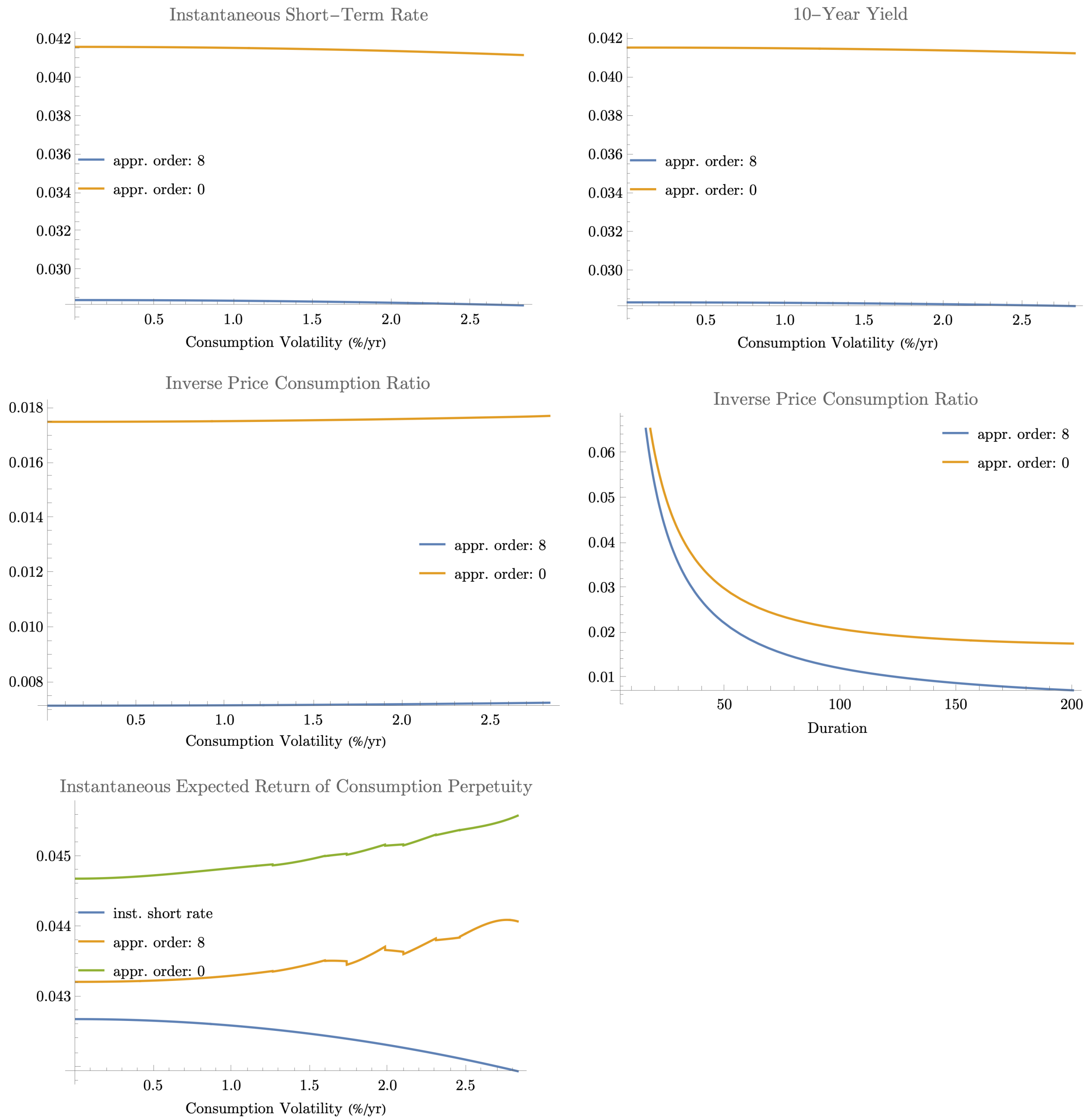
As a second application, I introduce the case when consumption volatility is time-varying. This is the result of setting $\sqrt{\sigma_{c1}} \neq= 0$, while consumption drift is constant because $\mu_{x1} = 0$. The state variable is also heteroskedastic, and it is guaranteed to be positive, because $\sigma_{x0} = 0$ and $\sqrt{\sigma_{x1}} \neq 0$. Figure 6 shows the case where $\epsilon = 0.1$, and all rates are not very sensitive to consumption volatility. Nevertheless, the figure demonstrates that the zeroth order approximation is roughly within 15 basis points compared to the higher approximation. Depending on the application, this difference could be considered negligible. In addition, while there is a difference in the level of the rates, the difference in the slope of the rates with respect to consumption volatility is not noticeable.



$\left\{\gamma{=}2,\ \rho{=}0.02,\ \mu_{c0}{=}0.0252,\ \sigma_{c0}{=}0,\ \sigma_{c1}{=}0.0004,\ \phi{=}0.92,\ \sigma_{ct}{=}\sqrt{\sigma_{c0}+\sigma_{c1}\,x_t},\ \sigma_{x1}{=}0.0169,\ \mu_{x0}{=}1,\ \rho\mathrm{cx}{=}\rho\rho_{cx}\ \&,\ \rho_{cx}{=}{-}0.5\right\}$

Figure 6: Variation: Time-varying consumption drift $-\ \epsilon = 0.1$
The first row shows the .

Figure 7 shows the case where $\epsilon = 0.7$. Now, the difference in the rates is certainly not negligible, as it ranges around 100 basis points. Nevertheless the slopes still appear the same. This is the result of the derivatives of K being very well approximated by the zeroth order approximation (Figure 9).



$\left\{\gamma=2,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\,x_t},\ \sigma_{x1}=0.0169,\ \mu_{x0}=1,\ \rho\text{cx}=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\right\}$

Figure 7: Variation: Time-varying consumption drift $-\ \epsilon = 0.7$
The first row shows the.

6 Conclusion

In conclusion, I have introduced a new method, based on perturbation theory, to express the value function when the agent exhibits recursive utility. The value function is expressed as a series in terms of ϵ and it constitutes a global perturbation solution. The value of ϵ is determined by the value of ψ, which represents the IES in the problem. The first term in the series (which multiplies ϵ^0) gives the solution for $\psi = 1$. Each further order of approximation only requires the solution of linear equations. Computing the first fifteen orders of approximation is relatively easy, but higher orders are typically computationally demanding as the number of coefficients increases by one for each order of approximation and the equations become increasingly complicated.

The method is useful for a wide range of calibrations. Tsai and Wachter (2018) only use the zeroth order approximation. I have shown that this can produce accurate results for a relatively low absolute value of ϵ, but the approximation can deteriorate as the absolute value of ϵ increases. Higher order approximations using my method can solve this issue, and this applies both for models with time-varying consumption drift and time-varying consumption volatility. The paper can also be extended to include multiple state variables and Poisson jump components in the consumption process. I have used the perturbation series to derive both the price of long-term bonds and the price consumption ratio of zero coupon consumption securities, consumption annuities and the consumption perpetuity. Furthermore, I have also derived the expected instantaneous return of the consumption perpetuity. Apart from being easy to implement, my method allows to easily check whether the results are accurate and how many orders of approximation are necessary for an acceptable solution. Despite not having derived exact convergence conditions, I have sketched the behaviour of the series for different orders of approximation, and I hope to derive more exact results in future versions of this paper.

For future research, it would be important to derive results that would guarantee the existence of a solution to this problem, as also discussed in Tsai and Wachter (2018), but such a task may not be easy. In addition, concentrating on my method, the perturbation series uniquely determines the value function, even if it is a divergent asymptotic series for some combinations of parameters and values of x. This means that further work following this approach, using more sophisticated mathematical analysis, could provide an expression of the solution that is easily computable and uniformly convergent, possibly in terms of special mathematic functions.[19]

[19] Applying a Padé approximation to the problem did not yield converging results in the regions that were diverging under the regular approximation.

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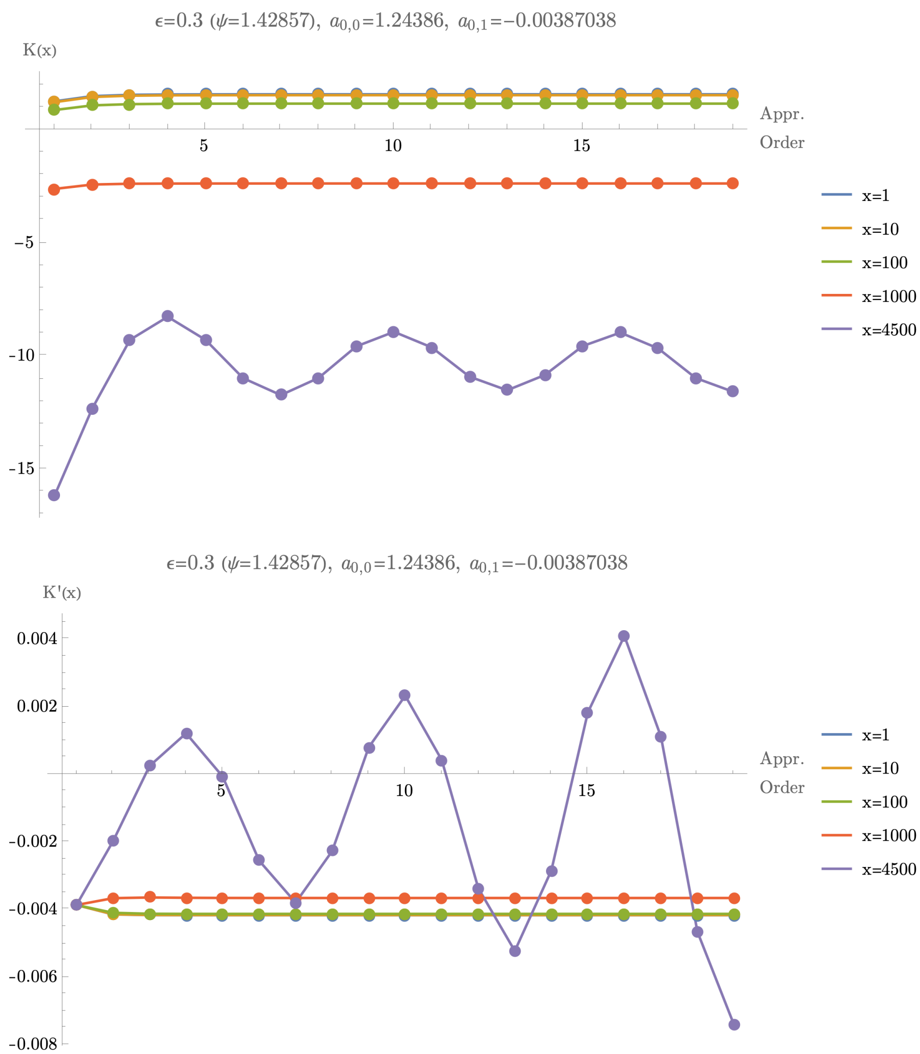
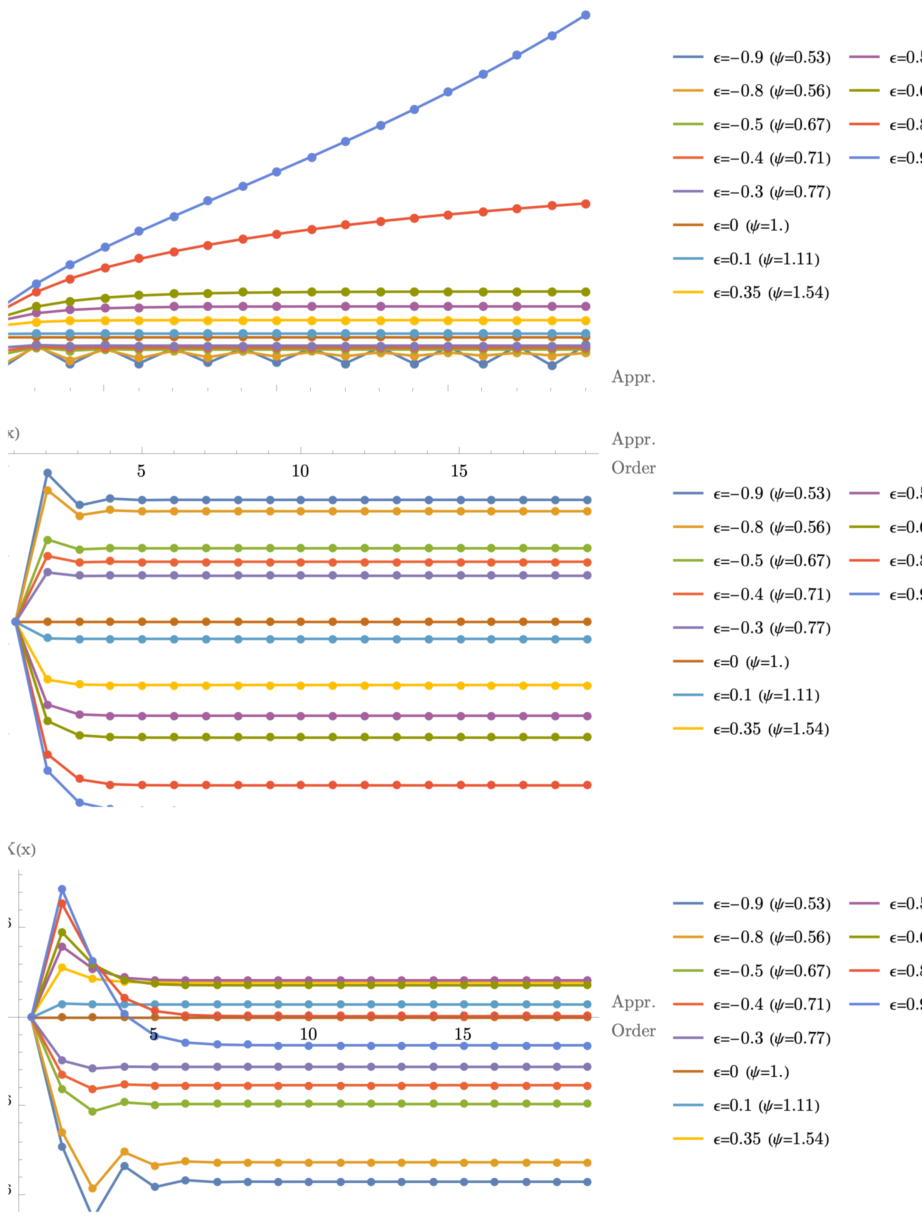


$\{\gamma=2,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\,x_t},\ \sigma_{x1}=0.0169,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 2: Variation: Time-varying consumption volatility.
This shows the convergence of the problem for different values of x.



$\{\gamma=2,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\,x_t}\ ,\ \sigma_{x1}=0.0169,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 3: Variation: Time-varying consumption volatility.
This shows the convergence of the problem for different values of ϵ. The plots correspond to the series of K and its two derivatives.

A Appendix

A.1 Proof of result in Equation (8)

The expression is derived from the Hamilton-Jacobi-Bellman equation, $\mathcal{D}V + f(C, V) = 0$, after the relevant quantities have been substituted. By applying Ito's Lemma to V, which is a function of C and x, the result is:

$$\frac{\mathcal{D}V}{V} = -\frac{1}{2}(\gamma - 1)\left(-\gamma\sigma_{\text{ct}}^2 + 2\mu_{\text{ct}} + \sigma_{\text{xt}}^2 K''(x) - \gamma\sigma_{\text{xt}}^2 K'(x)^2 + 2\mu_{\text{xt}} K'(x) + \sigma_{\text{xt}}^2 K'(x)^2\right) \tag{44}$$

Here, I can substitute the guessed expression for the value function, $V = \frac{C^{1-\gamma}e^{(1-\gamma)K(x)}}{1-\gamma}$, which I will verify later, in the previous expression and in the expression for flow utility:

$$\frac{\mathcal{D}V}{V} = (1-\gamma)\left(\mu_{\text{ct}} + \mu_{\text{xt}}K'(x) - \frac{\gamma\sigma_{\text{ct}}^2}{2} + \frac{(1-\gamma)\sigma_{\text{xt}}^2}{2}K'(x)^2 + \frac{\sigma_{\text{xt}}^2}{2}K''(x)\right)$$

$$f(C, V) = \frac{(1-\gamma)\rho V\left(\left(C((1-\gamma)V)^{-\frac{1}{1-\gamma}}\right)^{1-\frac{1}{\psi}} - 1\right)}{1 - \frac{1}{\psi}} = (1-\gamma)\rho\frac{\psi\left(1 - e^{\left(\frac{1}{\psi}-1\right)K[x]}\right)}{1-\psi} \tag{45}$$

After plugging these two expressions in the JHB equation, the result is:

$$\rho\frac{\psi\left(1 - e^{\left(\frac{1}{\psi}-1\right)K[x]}\right)}{1-\psi} + \mu_{\text{ct}} + \mu_{\text{xt}}K'(x) - \frac{\gamma\sigma_{\text{ct}}^2}{2} + \frac{(1-\gamma)\sigma_{\text{xt}}^2}{2}K'(x)^2 + \frac{\sigma_{\text{xt}}^2}{2}K''(x) = 0 \tag{46}$$

This is Equation (8) in the main text. By the fact that this is the result of the HJB equation, assuming that the solution exists, the guess is verified.

A.2 Derivation of the SDF with time-recursive utility

As mentioned in the paper the stochastic differential equation of the SDF can be derived based on the following expression:

$$\frac{d\Lambda}{\Lambda} = f_V(C, V)dt + \frac{df_C(C, V)}{f_C(C, V)} \tag{47}$$

So, flow utility is a central component of the derivation:

$$f(C,V) = \frac{\beta}{1-1/\psi}\big((1-\gamma)V\big)\left(\left(C((1-\gamma)V)^{-\frac{1}{1-\gamma}}\right)^{1-1/\psi} - 1\right) \tag{48}$$

The partial derivative of f with respect to V is:

$$f_V(C,V) = \frac{\rho\left((\gamma-1)\psi + (1-\gamma\psi)\left(C(V-\gamma V)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}\right)}{\psi-1} \tag{49}$$

The partial derivative of f with respect to C is:

$$f_C(C,V) = -\frac{(\gamma-1)\rho V\left(C(V-\gamma V)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}}{C} \tag{50}$$

As I implement Ito's Lemma directly using c_t and x_t as independent variables, I make the following replacements in the expressions above:

$$c_t = \log(C), \quad V = \frac{C^{1-\gamma}}{1-\gamma}e^{(1-\gamma)K(x_t)} \Rightarrow K(x_t) = \frac{\log\left(-\frac{C^{1-\gamma}}{(\gamma-1)V}\right)}{\gamma-1} \tag{51}$$

So, they become after simplification:

$$\begin{aligned} f_V(C,V) \to g(c_t,x_t) =& \frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi\right)}{1-\psi} \\ f_C(C,V) \to h(c_t,x_t) =& \rho e^{\left(\frac{1}{\psi}-\gamma\right)K(x_t) - c_t\gamma} \end{aligned} \tag{52}$$

And I implement Ito's Lemma on g_2. The partial derivatives are:

$$\begin{aligned}
\frac{\partial h(c_t,x_t)}{\partial c_t} =& \gamma\rho\left(-e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}\right) = -\gamma h(c_t,x_t) \\
\frac{\partial h(c_t,x_t)}{\partial x_t} =& \rho\left(\frac{1}{\psi}-\gamma\right)K'(x_t)\, e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \left(\frac{1}{\psi}-\gamma\right)K'(x_t)\, h(c_t,x_t) \\
\frac{\partial^2 h(c_t,x_t)}{\partial c_t^2} =& \gamma^2\rho e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \gamma^2 h(c_t,x_t) \\
\frac{\partial h(c_t,x_t)}{\partial x_t^2} =& \frac{\rho(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi^2} \\
=& \frac{(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)}{\psi^2}h(c_t,x_t) \\
\frac{\partial h(c_t,x_t)}{\partial c_t\partial x_t} =& \frac{\gamma\rho(\gamma\psi-1)K'(x_t)\, e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi} = \frac{\gamma(\gamma\psi-1)K'(x_t)\, h(c_t,x_t)}{\psi}
\end{aligned} \tag{53}$$

The expressions above should be plugged into the expression:

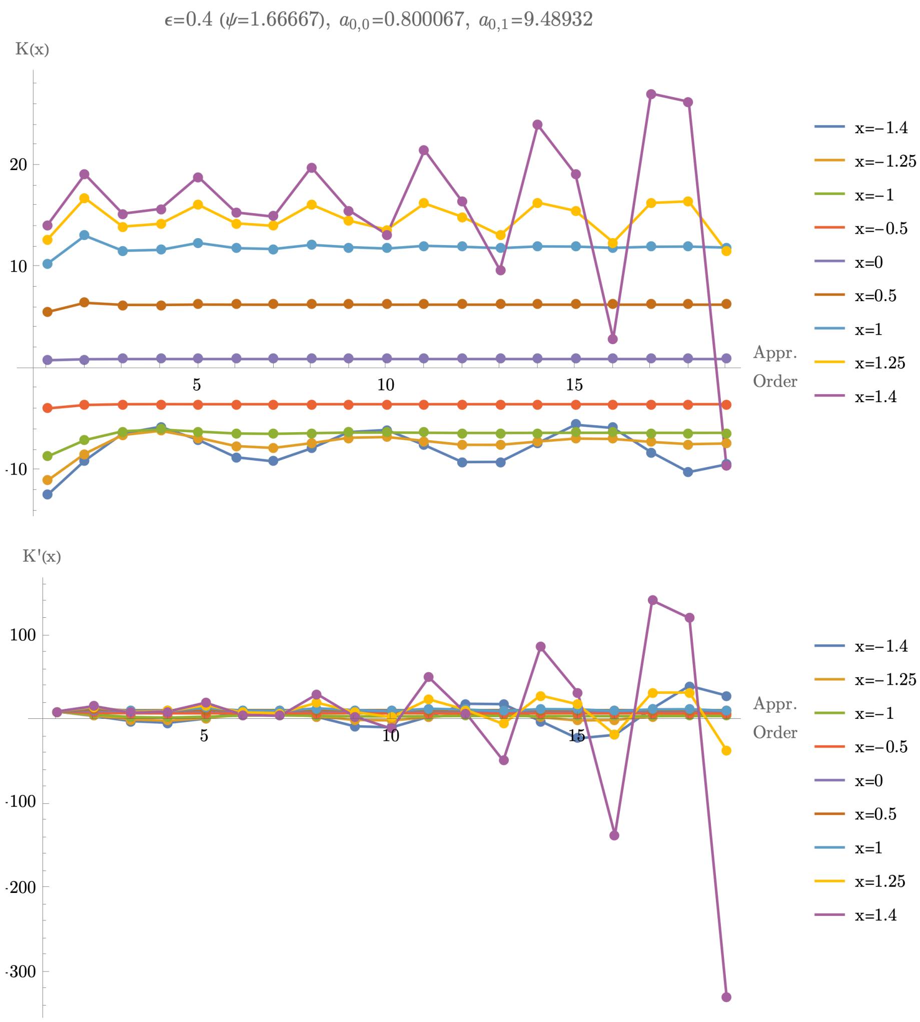
$$\begin{aligned}
\frac{df_C}{f_C} =& \Bigg(\frac{\partial h(c_t,x_t)}{\partial c_t}\mu_{ct} + \frac{\partial h(c_t,x_t)}{\partial x_t}\big(-\log(\phi)\big)(\mu_{x0}-x_t) \\
&+\frac{\sigma_{ct}^2}{2}\frac{\partial^2 h(c_t,x_t)}{\partial c_t^2} + \frac{\sigma_{xt}^2}{2}\frac{\partial^2 h(c_t,x_t)}{\partial x_t^2} + \frac{\rho_{cx}\sigma_{ct}\sigma_{xt}}{2}\frac{\partial^2 h(c_t,x_t)}{\partial c_t\partial x_t}\Bigg)dt \\
&+\frac{\partial h(c_t,x_t)}{\partial x_t}\sigma_{xt}dZ_{xt} + \frac{\partial h(c_t,x_t)}{\partial c_t}\sigma_{ct}dZ_{ct}
\end{aligned} \tag{54}$$

Then everything is plugged into Equation (47) to give the final result:

$$\begin{aligned}
\frac{d\Lambda}{\Lambda} =& \Bigg(\frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} + \frac{\gamma^2\sigma_{ct}^2}{2} - \gamma\mu_{ct} \\
&+\frac{(\gamma\psi-1)\Big(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t)+\sigma_{xt}^2\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)\Big)}{2\psi^2} \\
&+\frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}}-\gamma\psi+\psi\right)}{1-\psi}\Bigg)dt \\
&-\frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}dZ_{xt} - \gamma\sigma_{ct}dZ_{ct}
\end{aligned} \tag{55}$$

A.3 Convergence – Time-varying consumption drift

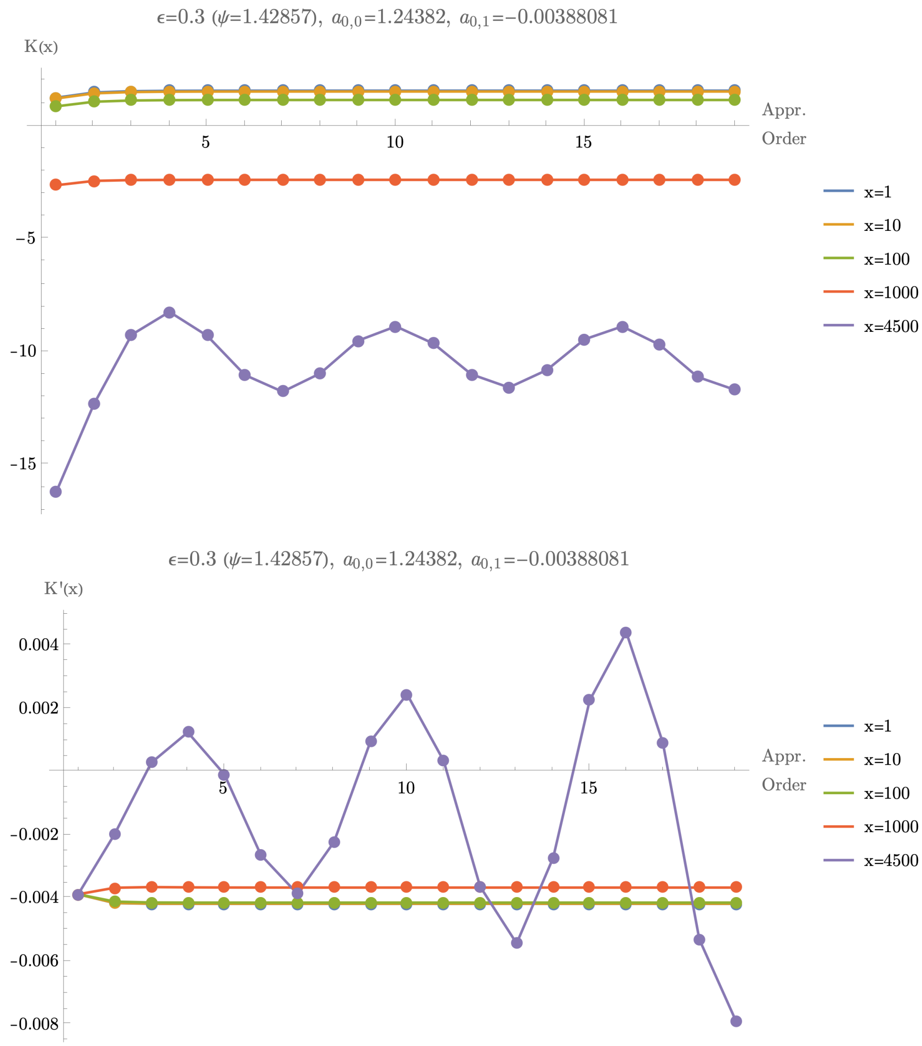
In the main paper I show convergence for the case, in which consumption volatility is time-varying. Here, I show the case when the consumption drift is time-varying. The convergence properties are similar in this case.



$\{\gamma=2,\ \rho=0.022,\ \phi=0.92,\ \rho_{cx}=0.5,\ \mu_{c0}=0.0252,\ \mu_{c1}=1,\ \mu_{x0}=0,\ \sigma_{c0}=0.0004,\ \sigma_{c1}=0,\ \sigma_{x0}=0.000159883,\ \sigma_{x1}=0\}$

Figure 8: Variation: Time-varying consumption drift.
This shows the convergence of the problem for different values of x.

In this case, the series converges for all values of ϵ between 0 and 1. this means that the approximation works quite effectively for all values of $\psi > 1$.



$\{\gamma=2,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\, x_t}\ ,\ \sigma_{x1}=0.16,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 9: Variation: Time-varying consumption drift.
This shows the convergence of the problem for different values of ϵ. The plots correspond to the series of K and its two derivatives.